

Fourier knots

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Abstract

We show that every knot has a checkerboard diagram and that every knot is the closure of a rosette braid. We define Fourier knots of type (n_1, n_2, n_3) as knots which have parametrizations where each coordinate function $x_i(t)$ is a finite Fourier series of length n_i , and conclude that every knot is a Fourier knot of type $(1, 1, n)$ for some natural number n .

1 Rosette braids

Let B_n be the braid group on n strings, $\pi : B_n \rightarrow S_n$ be the map to the symmetric group on n letters and $P_n = \ker(\pi)$ be the pure braid group on n strings. The generators of P_n are denoted by $A_{i,j}$. If π_0 is a permutation, then $k(\pi_0)$ denotes its number of cycles.

Definition 1.1

A braid of the form

$$\prod_{i=1}^n \left[\prod_{\substack{j \text{ odd} \\ j < s}} \sigma_j^{\varepsilon_{i,j}} \prod_{\substack{j \text{ even} \\ j < s}} \sigma_j^{\varepsilon_{i,j}} \right], \quad \varepsilon_{i,j} \in \{\pm 1\}$$

is called a *rosette braid of type (s, n)* . The set of rosette braids of type (s, n) is denoted by $\mathcal{R}(s, n)$.

Lemma 1.2

- a) i) $\alpha \in \mathcal{R}(s, 1) \Rightarrow k(\pi(\alpha)) = 1$.
- ii) $\alpha \in \mathcal{R}(s, s) \Rightarrow \pi(\alpha) = id$.
- iii) $\alpha \in \mathcal{R}(s, ns + 1) \Rightarrow k(\pi(\alpha)) = 1$.
- b) For each generator $A_{i,j}$ of P_n there is an $\alpha \in \mathcal{R}(s, s)$, so that $A_{i,j} = \alpha$.

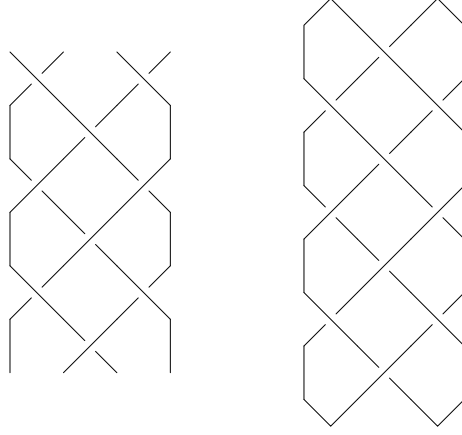


Figure 1: A rosette braid of type $(4, 3)$ and a checkerboard diagram of type $(4, 3)$

PROOF: a) Proposition i) is true for $s = 2$, because then the braid word has the form $\sigma^{\pm 1}$. An element of $\mathcal{R}(s, 1)$ is built out of an element of $\mathcal{R}(s - 1, 1)$ by a Markov-II-move (insertion of $\sigma_{s-1}^{\pm 1}$). Because the number of components is unchanged by a Markov-II-move, we conclude by induction that the proposition holds for all s .

ii) As shown in part i), the permutation $\pi(\alpha)$ of a braid $\alpha \in \mathcal{R}(s, 1)$ consists of one cycle. Hence the permutation of a braid in $\mathcal{R}(s, s)$ is the trivial permutation on s letters. Part iii) is an immediate consequence.

b) We consider the two strings i and j ($i < j$) of the braid $A_{i,j}$. If as above $\pi_1 = \pi(\alpha)$ is the permutation of a braid $\alpha \in \mathcal{R}(s, 1)$, then π_1 is a cycle of length s . Hence there is a k with $1 \leq k < s$, so that $\pi_1^k(i) > \pi_1^k(j)$, and thus the strings i and j cross each other. It is possible to arrange the strings in such a way that all strings but i and j can be pulled tight and the strings i, j form the generators $A_{i,j}$ or $A_{i,j}^{-1}$. \square

Theorem 1.3

Let $\alpha \in B_s$ be a braid, with closure a knot. Then α is conjugate to a rosette braid of type $(s, ns + 1)$ for a suitable n .

PROOF: Let $\alpha \in B_s$ be a braid, so that $\hat{\alpha}$ is a knot. Let δ be an arbitrary braid in $\mathcal{R}(s, 1)$ and π_1 its permutation. The permutations $\pi(\alpha)$ and π_1 are conjugate in the symmetric group because both consist of one cycle. Let $\beta \in B_s$ be a braid, so that $\pi(\beta)^{-1}\pi(\alpha)\pi(\beta) = \pi_1$. Then $\delta^{-1}\beta^{-1}\alpha\beta$ is a pure braid and because of Lemma 1.2 we can write it as an element of $\mathcal{R}(s, ns)$ for a suitable n . Multiplication with δ yields $\beta^{-1}\alpha\beta$ as an element of $\mathcal{R}(s, ns + 1)$. Hence we have shown that α is conjugate to a rosette braid of type $(s, ns + 1)$. \square

The braid index of a knot K is denoted by $br(K)$.

Corollary 1.4

Every knot K is the closure of a rosette braid with $br(K)$ strings. \square

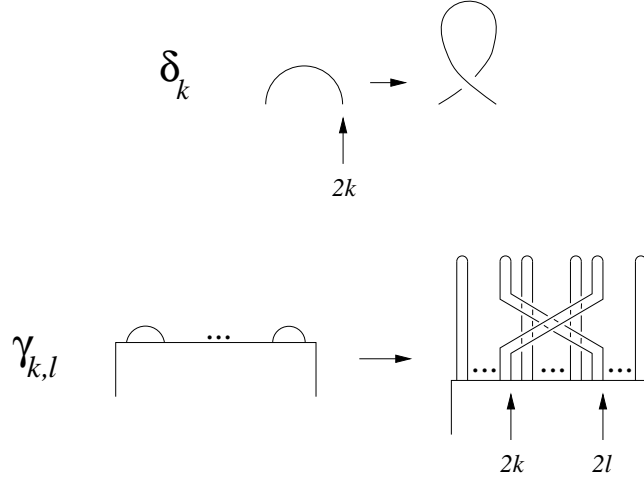


Figure 2: δ and γ to modify the permutation of a braid which constitutes a plat. The δ' and γ' are the mirrored operations for the lower plat closure.

2 Checkerboard diagrams

Definition 2.1

A knot diagram is called a *checkerboard diagram of type $(2b, n)$* , if it is the plat closure of a braid $\sigma_2^{\varepsilon_2} \dots \sigma_{2b-2}^{\varepsilon_{2b-2}} \cdot \alpha$ with $\alpha \in \mathcal{R}(2b, n)$ and $\varepsilon_2, \dots, \varepsilon_{2b-2} \in \{\pm 1\}$.

Let π_0 be the permutation of the braid $\sigma_2^{\varepsilon_2} \dots \sigma_{2b-2}^{\varepsilon_{2b-2}} \in B_{2b}$. The plat-operations δ, γ, δ' and γ' which we need for Lemma 2.2 are defined in Figure 2.

Lemma 2.2

Let K be a knot which is given as a plat closure of a braid $\alpha \in B_s$. Then there is a sequence of operations δ, δ', γ and γ' which transforms the plat $\bar{\alpha}$ to a plat $\bar{\beta}$ with $\pi(\beta) = \pi_0$.

PROOF: Using the operations of Figure 2, the permutation π_0 can be produced step by step. We start with string 1 and move its end-position to the position $\pi_0(1) = 1$. Then, travelling along the knot we can successively adjust the end-positions at the upper and lower plat-closure of the braid. The result is a plat with permutation π_0 . \square

Theorem 2.3

Every knot with bridge number b has a checkerboard diagram of type $(2b, 2nb)$ for a suitable n .

PROOF: In Lemma 2.2 we succeeded to represent the knot K as a plat $\bar{\alpha}$ with $\pi(\alpha) = \pi_0$. We consider the pure braid $\beta = \sigma_2 \sigma_4 \dots \sigma_{s-2} \cdot \alpha$. By Lemma 1.2 the braid β can be written as a rosette braid β' of type (s, ns) for some natural number n . Hence $\sigma_2^{-1} \sigma_4^{-1} \dots \sigma_{s-2}^{-1} \cdot \beta'$ is a checkerboard diagram for K . If we

$x_1(t) = \cos(2t + 0.8)$, $x_2(t) = \cos(3t + 0.15)$, $x_3(t) = \cos(4t + 1) + \cos(5t)$ for the figure-eight knot. Here we use the interval $t \in [0, 2\pi]$, in order to have the same parametrization as in [1].

Definition 3.5

If K is a knot, the *Fourier index* of K is the smallest number n for which K is a Fourier knot of type $(1, 1, n)$.

It is not known if there are knots with arbitrarily high Fourier index.

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